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THE SHELAH P-POINT INDEPENDENCE THEOREM

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ABSTRACT
In this paper, we present S. Shelah's example of a model of set theory in which there are no P-points in $\beta N \setminus N$. This settles the famous open question: "Is 'ZFC + there are no P-points in $\beta N \setminus N$' consistent?"

0. History

A famous open question was: "Is 'ZFC + there are no P-points in $\beta N \setminus N$' consistent?" Saharon Shelah has recently proven that "ZFC + there are no P-points in $\beta N \setminus N$" is consistent. This paper presents Shelah's solution, slightly modified by the author.

W. Rudin [6] has shown that the continuum hypothesis implies the existence of 2^c P-points. A similar argument can be used to show there are 2^c selective ultrafilters, assuming the continuum hypothesis. (All selective ultrafilters are P-points.) K. Kunen [3] has shown that there are no selective ultrafilters in the model of set theory obtained by adjoining $\aleph_2$ random reals to a model of set theory plus the continuum hypothesis. For a discussion of ultrafilters in general and P-points in particular, the reader is referred to [1].

The results and methods of proof in this paper are due to S. Shelah.

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Grigorieff [2] has generalized Sacks and Silver forcing to $P_{\mathscr{D}}$, where $\mathscr{D}$ is an ultrafilter on ω. He has shown that $P_{\mathscr{D}}$ preserves ω_1 iff $\mathscr{D}$ is a P-point.

In the model obtained by forcing with $(P_{\mathscr{D}})^\omega$, $\mathscr{D}$ cannot be extended to a

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P-point. This remains true even if $\mathcal{D}$ is only a P-filter, instead of a P-point. One way to try to destroy all P-points would be to force with a product of $P_{\mathcal{D}}$ for all P-filters. Unfortunately, this adds new P-points. However, there is a ccc suborder which does not add new P-points. We use Shelah's style of forcing to get a model of set theory in which we have the desired ccc suborder of $\Pi P_{\mathcal{D}}$. Another approach that might work is to assume $V = L$ and try to get a ccc suborder of $\Pi P_{\mathcal{D}}$ using combinatorial principles. However, in addition to seeming simpler, the method presented in this paper is more versatile, since by forcing instead of relaying on $V = L$ we have greater control over such things as $|2^{\aleph_0}|$.

The rest of this section is motivation. None of the rest of the material depends on what we do for the rest of section 0. Consequently, we make some assertions here without proof.

Let $\mathbf{Q} = \{g : \omega \times \omega \to 2 : \operatorname{dom}(g_i) \in I'\}$ where $g_i(k) = g(i,k)$ and I' is the ideal associated with a P-point P. Thus $A \in I'$ iff $\omega \setminus A \in P$. Let G be generic over V. We will show that there is no P-point $D \in V^{\mathbf{Q}}$ such that $P \subseteq D$. Suppose not.

Let $A_i = \{j : g_i(j) = 1$ for some $g \in G\}$. Let $A^0 = \omega \setminus A$ and $A^1 = A$. Define $\varepsilon : \omega \to D$ so that $A_i^{\varepsilon(i)} \in D$ where I is the ideal associated with D. Find two sequences i_n, j_n so that $\min\{i_{n+1}, j_{n+1}\} > \max\{i_n, j_n\}$ and so that $\varepsilon(i_n) = \varepsilon(j_n)$. Define $E_n = (A_{i_n}^0 \cap A_{j_n}^0) \cup (A_{i_n}^1 \cap A_{j_n}^1)$. Thus $E_n \in D$. Pick $B \in D$ so that $[0, f'(n)) \cup E_n \supseteq B$. Pick $g' \in \mathbf{P}$ which forces all that. Let $C \cup [0, f''(n)) \supseteq \operatorname{dom}(g'_n)$ where $C \in I'$. Define $f(n) = \max\{f''(i_n), f''(j_n), f'(n)\}$. It is possible to find such an f in V. Let $k \in [f(n), f(n+1))$. Define g so that it extends g' and so that $g(i_n, k) = 0$ and $g(j_n, k) = 1$.

We will now show $g \Vdash 0 = 1$. Let $k \in [f(n), f(n+1))$ and assume $k \notin C$. Then $g \Vdash k \notin B \setminus E_n$ since $f(n) \geqq f'(n)$. Since $g(i_n, k) = 0$ and $g(j_n, k) = 1$, $g \Vdash k \notin E_n$. Thus $g \Vdash k \notin B$. Since $n \geqq 0$ was arbitrary, $g \Vdash B \subseteq C \cup [0, f(0))$. But $g \Vdash B \in D$ & $C \in I$. Thus $g \Vdash 0 = 1$. This contradiction shows that the P-point P in $V^{\mathbf{Q}}$ can not be extended to a P-point D in $V^{\mathbf{Q}}$.

What we have just presented is a sketch of the proof of Lemma 6.4. We would like to defeat all P-points not just 1. Therefore, we need a sequence of P-points. So we will break the forcing up into 2 parts. The first part $\mathbf{P}$ will be designed to give us a partial order $\mathbf{Q}$ similar to the one above but now with the ability to defeat all the potential P-points not just 1. Both $\mathbf{P}$ and $\mathbf{Q}$ will need to be well-behaved. We intend to use Lemma 2.1 to show that every P-point in $(V^{\mathbf{P}})^{\mathbf{Q}}$ contains a P-filter in $V^{\mathbf{P}}$. For this we will need $\mathbf{Q}$ to be ccc. Therefore, our forcing for $\mathbf{P}$ must ensure that $\mathbf{Q}$ is ccc. We want to have a property on $\mathbf{P}$ to keep it well-behaved. $\mathbf{P}$ will turn out to be ω_1-complete and have the ω_2-cc.

We will now give some of the conditions that we will want to put on **P** and this motivates our definition of α-candidate (4.1). We will keep **P** "small" by requiring that every element be a countable subset of a set of cardinality ω_2. A Δ-system argument will give us that **P** has the ω_2-cc. We will want maximal anti-chains that we will keep countable to ensure that **Q** has the ccc (B-maximality). We will also want to do iterated forcing (in **P**) and we will want to keep the partial orders consistent with each other. Thus, we have the notions of reducing and B-bounding (see 4.1).

Using some of the important ideas in this paper, C. Mills [5] has given a simpler proof of the main result (6.5) that avoids technicalities of α-candidates.

I. Introduction

We begin with the convention that all our filters are over ω.

1.1. Definition. $A \subseteq^* B$ iff $A \setminus B$ is finite.

1.2. Definition. A *P-point* is a non-principal ultrafilter $\mathcal{D}$ over ω such that if A_n ($n \in \omega$) is a sequence of elements of $\mathcal{D}$, then there exists an $A \in \mathcal{D}$ such that $A \subseteq^* A_n$ for all $n \in \omega$.

1.3. Definition. If $\mathcal{D}$ is a filter, then let $\mathcal{I}_{\mathcal{D}} = \{A : \omega \setminus A \in \mathcal{D}\}$.

1.4. Definition. A filter $\mathcal{D}$ is *$\aleph_1$-saturated* iff there is no sequence A_α ($\alpha < \aleph_1$) such that

(1) $A_\alpha \subseteq \omega$,

(2) $A_\alpha \not\in \mathcal{I}_{\mathcal{D}}$,

(3) $A_\beta \cap A_\gamma \in \mathcal{I}_{\mathcal{D}}$ provided $\beta \neq \gamma$.

1.5. Definition. A *P-filter* is an $\aleph_1$-saturated filter $\mathcal{D}$ such that:

(1) Every cofinite set is in $\mathcal{D}$.

(2) If $\forall n \in \omega$ ($A_n \in \mathcal{D}$), then $\exists A \in \mathcal{D}\ \forall n \in \omega[A \subseteq^* A_n]$.

1.6. Definition. $\Vdash^{P}$ represents forcing for the partial order P.

1.7. Definition. $0 \Vdash^{\mathbf{P}}$ "a formula" is an abbreviation for $\forall p \in \mathbf{P}[p \Vdash^{\mathbf{P}}$ "a formula"].

II. Forcing facts

In this section, we present some general facts about forcing that will be used later. When we write $p_1 \geqq p_2$ for conditions we mean p_1 extends p_2. Thus $p_1 \geqq p_2$

and p_1 in the generic set implies p_2 is in the generic set. Thus, p_1 forces more things than p_2.

The next lemma tells us that if we force with a ccc partial order $\mathbf{Q}$ and every "new" function from ω to ω is bounded by an "old" one, then every P-point in $V^{\mathbf{Q}}$ contains a P-filter which is in V.

2.1. LEMMA. Hypothesis:

(i) *$\mathbf{Q}$ is a ccc partial order.*

(ii) $0 \Vdash^{\mathbf{Q}}$ "*τ is a P-point*".

(iii) *Every function (from ω to ω) in $V^{\mathbf{Q}}$ is bounded by a function in V.*

(iv) *Define* $\mathcal{D}^\tau = \{A \subseteq \omega : 0 \Vdash^{\mathbf{Q}} A \in \tau\}$.

Conclusion:

$\mathcal{D}^\tau$ is a P-filter in V.

PROOF. Clearly, $\mathcal{D}^\tau$ is a filter containing every cofinite set. Suppose $\mathcal{D}^\tau$ is not $\aleph_1$-saturated. Hence, we have a sequence $\langle A_\alpha : \alpha < \aleph_1 \rangle$ in V contradicting $\aleph_1$-saturation. Thus, $\omega \setminus A_\alpha \notin \mathcal{D}^\tau$. Hence, it is not the case that $0 \Vdash^{\mathbf{Q}} \omega \setminus A_\alpha \in \tau$. Thus, $p_\alpha \Vdash^{\mathbf{Q}} \omega \setminus A_\alpha \notin \tau$ for some $p_\alpha \in \mathbf{Q}$. Thus, $p_\alpha \Vdash^{\mathbf{Q}} A_\alpha \in \tau$, since τ is an ultrafilter. Thus, p_α, p_β are incompatible for $\alpha \neq \beta$, contradicting ccc. Therefore, we have shown that $\mathcal{D}^\tau$ is $\aleph_1$-saturated.

Now assume we have a sequence A_n ($n \in \omega$) in V with all $A_n \in \mathcal{D}^\tau$. There are names $\boldsymbol{A}$, $\boldsymbol{f}$ in the forcing language for $\mathbf{Q}$ such that $0 \Vdash^{\mathbf{Q}} \boldsymbol{A} \in \tau \;\&\; \boldsymbol{A} \subseteq A_n \cup [0, \boldsymbol{f}(n))$. We can find a countable set $\{p_i : i \in \omega\}$ of conditions maximal with respect to:

(1) If $i \neq j$, then p_i, p_j are incompatible.

(2) $p_i \Vdash^{\mathbf{Q}} \forall n \in \omega [\boldsymbol{f}(n) \leqq g_i(n)]$ for some $g_i \in V$.

Every $\mathbf{Q}$-generic set contains exactly one of the p_i's. Define $h(n) = \max\{g_0(n), \cdots, g_n(n)\}$. Define $A = \bigcap_{n \in \omega} [A_n \cup [0, h(n))]$. Thus, $A \in V$ and $A \subseteq^* A_n$ for all $n \in \omega$. We also have the following:

$$p_i \Vdash A_0 \cup [0, h(0)) \in \tau \;\&\; \cdots \;\&\; A_{i-1} \cup [0, h(i-1)) \in \tau \;\&\; \bigcap_{\substack{n \in \omega \\ n \geqq i}} A_n \cup [0, h(n)) \in \tau,$$

$$p_i \Vdash \bigcap_{n \in \omega} A_n \cup [0, h(n)) \in \tau,$$

$$p_i \Vdash A \in \tau.$$

Since each generic set contains a p_i, we have that $0 \Vdash A \in \tau$. Thus, $A \in \mathcal{D}^\tau$ and $A \subseteq^* A_n$ for all $n \in \omega$.

Thus, $\mathcal{D}^\tau$ is a P-filter. $\square$

2.2. CONVENTION. Whenever we say "increasing" we do *not* mean "strictly increasing". For example, we consider the constant sequence to be increasing.

2.3. REMARKS. An $\aleph_1$-complete partial order is one where every increasing sequence p_n $(n \in \omega)$ of conditions has an upper bound.

If we have sequence E_n $(n \in \omega)$ of elements of $\mathcal{I}_{\mathcal{D}}$ where $\mathcal{D}$ is a P-filter, then there exists an $E \supseteq^* E_n$ $(n \in \omega)$ such that $E \in \mathcal{I}_{\mathcal{D}}$. $E_n \setminus E$ is finite, but it might be a large finite set. The next lemma says we can extract an infinite subsequence such that $|E_n \setminus E|$ is bounded on the subsequence by a preassigned function that increases to infinity.

2.4. COLLECTING LEMMA. Hypothesis:

(1) $\mathbf{P}$ *is an* $\aleph_1$*-complete partial order,*

(2) S_0 *is an infinite subset of* ω,

(3) $h \in \mathbf{P}$ *and* $h \Vdash$ "τ *is a* P*-filter*",

(4) E_n $(n \in \omega)$ *is an increasing sequence of subsets of* ω *such that* $h \Vdash E_n \in \mathcal{I}_\tau$ *for all* $n \in \omega$,

(5) c_n *is an increasing sequence of finite subsets of* ω *and* $\bigcup_{n\in\omega} c_n = \omega$.

Conclusion:

For each $n_0 \in S_0$ *there exists an* $h' \in \mathbf{P}$, *an infinite* $S_1 \subseteq S_0$, *and an* $E' \subseteq \omega$ *such that*

(1) $h' \geqq h$,

(2) $S_1 \supseteq S_0 \cap [0, n_0]$,

(3) $E' \supseteq E_{n_0}$,

(4) $h' \Vdash E' \in \mathcal{I}_\tau$,

(5) $\forall n \in S_1 [E' \supseteq E_n \setminus c_n]$.

PROOF. Since τ is a P-filter and $\mathbf{P}$ is $\aleph_1$-complete, we get an $h_1 \in \mathbf{P}$ such that $h_1 \geqq h$ and $h_1 \Vdash E^0 \in \mathcal{I}_\tau$ for some $E^0 \in V \cap 2^\omega$ such that $E^0 \supseteq^* E_n$ for all $n \in \omega$. Define $f : \omega \to \omega$ so that:

(i) $E^0 \cup c_{f(n)} \supseteq E_n$.

(ii) f is strictly increasing.

(iii) Image$(f) \subseteq S_0$.

Define a sequence t_n $(n \in \omega)$ by letting $t_0 = n_0$ and $t_{i+1} = f(t_i)$ for $i \geqq 0$. Let $\langle R_\alpha : \alpha < \aleph_1 \rangle$ be a sequence of infinite, almost disjoint subsets of $\{t_i : i \in \omega\}$. Let

$$e_\alpha = E^0 \cup \bigcup_{t_i \in R_\alpha} (c_{t_{i+1}} \setminus c_{t_i}).$$

Thus $h_1 \Vdash e_\alpha \cap e_\beta \in \mathcal{I}_\tau$ provided $\alpha \neq \beta$.

By $\aleph_1$-saturation of τ, there exists an $h_2 \in \mathbf{P}$ such that $h_2 \geqq h_1$ and $h_2 \Vdash e_{\alpha_0} \in \mathcal{I}_\tau$ for some $\alpha_0 \in \omega_1$. Let $E^2 = e_{\alpha_0}$ and $S_2 = R_{\alpha_0}$. If $t_1 \in S_2$, then

$$E^2 \supseteq E^0 \cup (c_{t_{i+1}} \setminus c_{t_i}) \supseteq E_{t_i} \setminus c_{t_i}.$$

The lemma follows by letting $h' = h_2$, $E' = E^2 \cup E_{n_0}$, and $S_1 = S_2 \cup (S_0 \cap [0, n_0])$.

□

III. Notation

In this section g will be a partial function from $\alpha \times \omega$ into 2 with a countable domain. A, B are countable collections of such g's. For each g and ordinal i, we have a partial function $f^i_g : \omega \to \{0,1\}$ defined by $f^i_g(n) = g(i,n)$ for all $n \in \omega$. For each ordinal β, $A \restriction (\beta \times \omega) = \{g \restriction \beta \times \omega : g \in A\}$, and similarly for B. We denote by TOP(g) the least ordinal i such that $g \restriction (i \times \omega) = g$. We denote by CEIL$(A)$ the least ordinal i such that $\forall g \in A(\mathrm{TOP}(g) \leqq i)$. We let $\mathrm{FI}(g) = \{\beta : \exists n(\langle \beta, n\rangle \in \mathrm{DOM}(g))\}$. (FI stands for first.) We let $\mathrm{FI}(A) = \bigcup_{g \in A} \mathrm{FI}(g)$. We define CEIL$(B)$ and FI(B), just like we defined CEIL(A) and FI(A). We say that $g_1 =^* g_2$ iff there is a finite set c such that

$$g_1 \restriction (\mathrm{DOM}(g_1) \setminus c) = g_2 \restriction (\mathrm{DOM}(g_2) \setminus c).$$

Finally, let $\mathcal{D}_\beta$ $(\beta < \gamma)$ be a sequence of filters. Then we let

$$Q_{\langle \mathcal{D}_\beta : \beta < \gamma\rangle} = \{g : G : \gamma \times \omega \xrightarrow{\text{partial}} \{0,1\},\ |\mathrm{DOM(g)}| \leqq \aleph_0, \text{ and}$$
$$\forall i < \gamma[\mathrm{DOM}(f^i_g) \in \mathcal{I}_{\mathcal{D}_i}]\}.$$

Thus, $Q_{\langle \beta : \beta < \gamma\rangle}$ is the set of partial functions g each of whose "sections" f^i_g are "small".

IV. Candidates

In the next section we will define partial orders $\mathbf{P}_\alpha$ $(\alpha \leqq \omega_2)$ and force with these. Before defining $\mathbf{P}_\alpha$, we need to define a superset of $\mathbf{P}_\alpha$, the set of α-candidates. In this section, we introduce candidates and prove some rather technical facts about them. We will need these technical results later when we force, but in this section we do no forcing and our results do not depend on forcing in any way.

4.1. Definition of α-candidate. h is a *α-candidate* iff DOM(h) is a countable subset of $\alpha + 1$, $0 \in \mathrm{DOM}(h)$, $h(i) = (A^h_i, F^h_i)$ for all $i \in \mathrm{DOM}(h)$, and the following conditions are satisfied for all $i \in \mathrm{DOM}(h)$:

(0) A_i^h is a non-empty, countable collection of functions into $\{0, 1\}$ each having a domain which is a countable subset of $i \times \omega$. F_i^h is a non-empty, countable collection of subsets of A_i^h.

(1) If $g \in A_i^h$, $g' = {}^*g$, and $\mathrm{DOM}(g') \subseteq \mathrm{FI}(A_i^h) \times \omega$, then $g' \in A_i^h$.

(2) $\mathrm{FI}(A_i^h) = \{j : j < i \text{ and } j + 1 \in \mathrm{DOM}(h)\}$.

(3) $\mathrm{DOM}(h)$ is closed under sup's.

(4) If $\beta \in \alpha + 1$, $g \in A_i^h$, then $g \restriction (\beta \times \omega) \in A_i^h$.

(5) [B-maximality] if $B \in F_i^h$, then every member of A_i^h can be extended to a member of A_i^h which extends some member of B.

(6) [A-union] if $g_1, g_2 \in A_i^h$ and $g_1 \cup g_2$ is a function, then $g_1 \cup g_2 \in A_i^h$.

(7) [Reducing] Let $j < i$, $i, j \in \mathrm{DOM}(h)$. Then we have $F_j^h \subseteq F_i^h$, $F_j^h = \{B \restriction (j \times \omega) : B \in F_i^h\}$, and $A_j^h = \{g \restriction (j \times \omega) : g \in A_i^h\}$.

(8) [B-bounding] Fix $g \in A_i^h$. Let $B \in F_i^h$, $j \leqq i$, $j, i \in \mathrm{DOM}(h)$. Then $B_j^g \in F_j^h$ where we define

$$B_j^g = \{g_1 \in A_j^h : g_1 \cup g \text{ is not a function or}$$

$$\exists g_2 \in A_i^h (g_2 \restriction (j \times \omega) = g_1 \text{ and } g_2 \supseteq g \text{ and } g_2 \text{ is above some member of } B)\}.$$

4.2. Remarks. Since the definition of α-candidate is so long, some remarks about it are appropriate. (1) says that A_i^h is closed under finite changes. (2) says that if $\langle j, n \rangle \in \mathrm{DOM}(g)$ for some $g \in A_i^h$, then $j + 1 \in \mathrm{DOM}(h)$. (4) says that A_i^h is closed under restricting the domain. Notice that (4) implies $\varnothing \in A_i^h$. If every $i \in \mathrm{DOM}(h)$ happens to be a limit ordinal, (2) tells us that $A_i^h = \{\varnothing\}$. The B's are supposed to represent maximal "antichains". (5) assures us that the B's will be maximal, but the B's need not in general be antichains. If $\beta > \alpha$ and h is an α-candidate, then h is also a β-candidate. By (4) and (7), if $j < i$, then $A_j^h \subseteq A_i^h$.

Let $j \leqq i$. Fix $g \in A_i^h$ and $B \in F_i^h$. B-bounding, together with (5), implies that every element of A_j^h can be extended to another element of A_j^h that is either incompatible with g or else is the restriction to $j \times \omega$ of an element of A_i^h that is an extension of both g and some element of B.

In fact, as we will see in the proof of Proposition 4.5, an α-candidate already satisfies this condition with requiring B-bounding. However, we will need B-bounding when we glue candidates together in the "gluing lemma" (4.13).

4.3. Definition. $A^h = \bigcup_{i \in \mathrm{DOM}(h)} A_i^h$; $F^h = \bigcup_{i \in \mathrm{DOM}(h)} F_i^h$.

4.4. Remark. By (3), if $g \in A^h$, then $\mathrm{TOP}(g) \in \mathrm{DOM}(h)$. Thus, $\mathrm{CEIL}(A^h) \in \mathrm{DOM}(h)$. Also, if $B \in F_\beta^h$, then $B \restriction (\gamma \times \omega) \in F_\beta^h$ for $\gamma < \beta$.

If $\gamma \in \mathrm{DOM}(h)$ and $\gamma \geqq \mathrm{CEIL}(A^h)$, then $A^h_\gamma = A^h$ and $F^h_\gamma = F^h$. In particular, $A^h_{\mathrm{CEIL}(A^h)} = A^h$ and $F^h_{\mathrm{CEIL}(A^h)} = F^h$. If $\gamma \in \mathrm{DOM}(h)$ and $\mathrm{CF}(\gamma) \geqq \aleph_1$, then $A^h_\gamma = A^h_\beta$ for some $\beta \in \gamma \cap \mathrm{DOM}(h)$.

Notice that (A^h, F^h) essentially determines the candidate. The only thing (A^h, F^h) does not determine is $\mathrm{DOM}(h)$. For example, F^h_β consists exactly of all (countable) sets of the form B $(\beta \times \omega)$ where B is any element of F^h. Thus, if $\beta < \gamma$, F^h_β consists exactly of all (countable) sets of the form B $(\beta \times \omega)$ where B is any element of F^h. Also, $A^h_\beta = A^h \restriction (\beta \times \omega)$.

Our main goal in this section is to show how we can glue candidates together. Our next two propositions tell us that when we glue candidates we do not need to worry about B-bounding. The first of the two is weaker than the second, but is needed in its proof.

4.5. Proposition. Hypothesis:

(1) *h is an α-candidate (except for B-bounding),*

(2) *$j, i \in \mathrm{DOM}(h)$ and $j \leqq i$,*

(3) *$g \in A^h_i$ and $B \in F^h_i$.*

Conclusion: *There exists an α-candidate (except for B-bounding) such that:*

(1) $\mathrm{DOM}(h_1) = \mathrm{DOM}(h)$,

(2) *for all $i_0 \in \mathrm{DOM}(h)$, we have that $A^h_{i_0} = A^{h_1}_{i_0}$, $F^h_{i_0} \subseteq F^{h_1}_{i_0}$.*

(3) $B^g_j \in F^{h_1}_j$.

Proof. Define h_1 so that $\mathrm{DOM}(h_1) = \mathrm{DOM}(h)$ and for all $i_0 \in \mathrm{DOM}(h)$ we have $A^{h_1}_{i_0} = A^h_{i_0}$ and

$$F^{h_1}_{i_0} = F^h_{i_0} \cup \{B^g_j \restriction (k \times \omega) : k \leqq i_0\}.$$

Everything is clear except, perhaps, that B-maximality holds. Clearly, B-maximality holds for any $B \in F^h_{i_0}$. First we check B-maximality for $i_0 \leqq j$. Thus $B_0 \in F^{h_1}_{i_0} \setminus F^h_{i_0}$ is of the form $B^g_j \restriction (k \times \omega)$ for some $k \leqq i_0$. Let $g_1 \in A^{h_1}_{i_0}$. If $g \cup g_1$ is not a function, then $g_1 \restriction (k \times \omega) \in B_0$. So without loss of generality, assume $g \cup g_1$ is a function. Since $g \cup g_1 \in A^h_i$, there is a $g_2 \in A^h_i$ such that $g_2 \supseteq g \cup g_1$ and g_1 is above some member of B. Since $g_2 \restriction (k \times \omega) \in B_0$, $g_2 \restriction (i_0 \times \omega) \in A^{h_1}_{i_0}$ is above some member of B_0. But $g_2 \restriction (i_0 \times \omega) \supseteq g_1$. This shows B-maximality for $i_0 \leqq j$.

We now check B-maximality for $i_0 > j$. Thus, $B_0 \in F^{h_1}_{i_0} \setminus F^h_{i_0}$ is of the form $B^g_j \restriction (k \times \omega)$ for some $k \leqq j$. Let $g_1 \in A^{h_1}_{i_0}$. Thus $g_1 \restriction (j \times \omega) \in A^{h_1}_j$. By the case for $i_0 = j$, we get a $g_2 \in A^{h_1}_j$ such that $g_2 \supseteq g_1 \restriction (j \times \omega)$ and g_2 is above some member of B_0. Therefore, $g_2 \cup g_1 \in A^{h_1}_{i_0}$ and $g_2 \cup g_1$ is above some member of B_0. $\square$

4.6. PROPOSITION. *For every α-candidate (except for B-bounding) h there exists an α-candidate h_1 such that* $\mathrm{DOM}(h) = \mathrm{DOM}(h_1)$ *and for all* $i \in \mathrm{DOM}(h)$ *we have that* $A_i^h = A_i^{h_1}$ *and* $F_i^h \subseteq F_i^{h_1}$.

PROOF. Follows from repeated applications of the previous proposition. □

4.7. DEFINITION (of a partial order on α-candidates). Let h_1, h_2 be α-candidates. $h_1 \leqq h_2$ iff $\mathrm{DOM}(h_1) \subseteq \mathrm{DOM}(h_2)$ and for all $i \in \mathrm{DOM}(h_1)$ we have that $A_i^{h_1} \subseteq A_i^{h_2}$ and $F_i^{h_1} \subseteq F_i^{h_2}$.

4.8. DEFINITION. $\langle p_n, c_n\rangle$ $(n \in \omega)$ denoted by $\overrightarrow{\langle p_n, c_n\rangle}$ is a *covering sequence* for an α-candidate h iff

1. c_n $(n \in \omega)$ is an increasing sequence of finite sets such that $\bigcup_{n\in\omega} c_n = \mathrm{FI}(A^h) \times \omega$.
2. p_n $(n \in \omega)$ is an increasing sequence of elements of A^h.
3. $\mathrm{DOM}(p_n) \supseteq c_n$.
4. For all $B \in F^h$, there exists an $n_B \in \omega$ such that for all $n \geqq n_B$, we have that if $\mathrm{DOM}(p') = \mathrm{DOM}(p_n)$, $p' \in A^h$, and $p' \restriction [\mathrm{DOM}(p') \setminus c_n] = p_n \restriction [\mathrm{DOM}(p_n) \setminus c_n]$, then p' is above some member of B.

4.9. REMARKS. From now on, if $\mathrm{DOM}(p') = \mathrm{DOM}(p)$ and

$$p' \restriction [\mathrm{DOM}(p') \setminus c] = p \restriction [\mathrm{DOM}(p) \setminus c],$$

we will say p' is "p changed on a subset of c".

We now show how covering sequences can be constructed.

4.10. PROPOSITION. Hypothesis:

(1) $j \in \mathrm{DOM}(h)$, *where h is an α-candidate,*

(2) $B_1, \cdots, B_n \in F_j^h$,

(3) $c \subseteq \mathrm{FI}(A_j^h) \times \omega$ *and c is finite,*

(4) $p \in A_j^h$.

Conclusion:

There exists a $p' \in A_j^h$ such that

(1) $p' \supseteq p$ *and* $\mathrm{DOM}(p') \supseteq c$,

(2) $\forall i \leqq n$ [*if p'' is p' changed on a subset of c, then p'' is above some member of* B_i].

PROOF. By induction, we may assume that $n = 1$. Let $\{s_1, \cdots, s_k\} = 2^c$ and let $p_0 = p$. For $i \geqq 0$, define p_{i+1} recursively so that $\mathrm{DOM}(p_{i+1}) \supseteq c$, $p_{i+1} \supseteq p_i$ and $p_{i+1} \in A_j^h$ and $(p_{i+1} \restriction [\mathrm{DOM}(p_{i+1}) \setminus c]) \cup s_{i+1}$ is above some member of B_1. B-

maximality and closure of A_j^h under finite changes guarantee that p_{i+1} exists. The proposition follows from $p' = p_k$. □

4.11. PROPOSITION. *For any $g \in A^h$ where h is an α-candidate, there exists a covering sequence $\overrightarrow{\langle p_n, c_n\rangle}$ such that $g \subseteq p_0$.*

PROOF. Let $B_1, \cdots, B_n, \cdots$ be all the elements of F^h. Let c_n be an increasing sequence of finite sets such that $\bigcup_{n\in\omega} c_n = \mathrm{FI}(A^h) \times \omega$. Let $p_{-1} = g$.

Define p_i $(i \geqq 0)$ recursively by requiring that $p_i \supseteq p_{i-1}$ and $\mathrm{DOM}(p_i) \supseteq c_i$ and $p_i \in A^h$. We further require that if p_i is changed on a subset of c_i, then it is above some member of B_j $(j \leqq i)$. □

4.12. DEFINITION. Let $\overrightarrow{\langle p_n, c_n\rangle}$ be a covering sequence for the α-candidate h. Let $g : \beta \times \omega \xrightarrow{\text{partial}} \{0,1\}$. We say that *$g$ is over $\overrightarrow{\langle p_n, c_n\rangle}$ on S for β* iff

(1) S is an infinite subset of ω,

(2) $\forall n \in S[g \supseteq p_n \restriction ((\mathrm{DOM}(p_n)\setminus c_n) \cap (\beta \times \omega))]$.

Our next lemma tells us how to glue candidates together. It is quite general, and we will rarely need its full strength. Sometimes, for example, we will not be interested in a sequence h_n^1 $(n \in \omega)$ but in just one h^1. We can apply the lemma with the sequence h_n^1 $(n \in \omega)$ equal to the constant sequence h^1. Other times, we will not be interested in g or the covering sequence. In this case, we take $\gamma^3 = 0$. Unfortunately, due to its generality and the fact that it tells us how to glue candidates together, its statement and proof are somewhat technical. Essentially the lemma tells us that when we glue candidates together a typical element of $A^{h_1^2}$ (the glued together candidate) will be of the form $g_1 \cup g_2 \cup g_3$ (provided the union is a function) where g_1 is in some $A^{h_n^1}$ for an increasing sequence of α_n-candidates h_n^1, g_3 is "almost equal" to the designated function g which is over the given covering sequence, and g_2 is in A^{h_2}. We require that $h_2 \restriction (\gamma_n^1 + 1) \leqq h_n^1$.

4.13. GLUING LEMMA. Hypothesis:

(1) *γ_n^1 $(n \in \omega)$ is a sequence of ordinals increasing to γ^1.*

(2) *h_n^1 is a γ_n^1-candidate (for all $n \in \omega$).*

(3) *h_n^1 $(n \in \omega)$ is an increasing sequence of γ^1-candidates.*

(4) *$\gamma^2 \geqq \gamma^1$ and $\gamma^2 \geqq \gamma^3$.*

(5) *h^2 is an γ^2-candidate.*

(6) *$h^2 \restriction (\gamma_n^1 + 1) \leqq h_n^1$ for all $n \in \omega$.*

(7) *$\overrightarrow{\langle p_n, c_n\rangle}$ is a covering sequence for h^2.*

(8) *g is over $\overrightarrow{\langle p_n, c_n\rangle}$ on S for γ^3.*

(9) *$g \restriction (\gamma_n^1 \times \omega) \in A^{h_n^1}$ for all $n \in \omega$.*

(10) $\mathrm{FI}(g) \subseteq \mathrm{FI}[A^{h^2}) \cup \bigcup_{n\in\omega} \mathrm{FI}(A^{h_n^1})$.

Conclusion:

There exists a γ^2*-candidate* h_1^2 *such that*

(1) $h_1^2 \geqq h^2$ *and* $h_1^2 \geqq h_n^1$ *for all* $n \in \omega$.

(2) $\mathrm{DOM}(h_1^2) = \mathrm{DOM}(h^2) \cup \bigcup_{n \in \omega} \mathrm{DOM}(h_n^1)$ *closed under* sup's.

(3) *For all* $i \in \mathrm{DOM}(h_1^2)$,

$$A_i^h = \{g_1 \cup g_2 \cup g_3 : g_1 \cup g_2 \cup g_3 \text{ is a function and where}$$

$$g_1 \in A^{h_n^1} \restriction (i_1 \times \omega) \text{ for some } n \in \omega,\ i_1 \leqq i;$$

$$g_2 \in A^{h^2} \restriction (i_2 \times \omega) \text{ for some } i_2 \leqq i;$$

$$g_3 = g^3 \restriction (i_3 \times \omega) \text{ for some } i_3 \leqq i \text{ where}$$

$$g^3 = {}^*g \text{ and } \mathrm{FI}(g^3) \subseteq \mathrm{FI}(g) \cup (\gamma^3 \cap \mathrm{FI}(A^{h_2}))\}.$$

Proof. Let $\mathrm{DOM}(h_1^2)$ be defined as in (2) of the conclusion. Define $A_i^{h_1^2}$ as in (3) of the conclusion. Define $F_i^{h_1^2} = \{B \restriction (i \times \omega) : B \in F^{h^2}$ or $B \in F^{h_n^1}$ for some $n \in \omega\}$. Assume (*) h_1^2 is a γ^2-candidate (except for B-bounding). Notice that any one or all of g_1, g_2, g_3 could be the empty function.

The result follows from (*) by Proposition 4.6. We now prove (*). The main difficulty is B-maximality, which we will check first. Let $B_0 \in F_i^{h_1^2}$, $g_0 \in A_i^{h_1^2}$ where $i \in \mathrm{DOM}(h_1^2)$. g_0 has the form $g_1 \cup g_2 \cup g_3$ where g_1, g_2, g_3 are given in the conclusion of the lemma.

Case 1. $B_0 \in F^{h_n^1}$ for some $h_{n_0}^1$, $n \in \omega$.

Pick i_0, n_0 so that $g_1 \in A_{i_0}$ and $B_0 \in F_{i_0}^{h_{n_0}^1}$. Thus, $g_0 \restriction (i_0 \times \omega) \in A_{i_0}^{h_{n_0}^1}$. There exists a $g_0^1 \in A_{i_0}^{h_{n_0}^1}$ such that $g_0^1 \supseteq g_0 \restriction (i_0 \times \omega)$ and g_0^1 is above some member of B_0. Let $g_1^1 = g_0^1 \restriction (i \times \omega)$. Thus $g_1^1 \cup g_0 \in A_i^{h_1^2}$ and $g_1^1 \cup g_0$ is above some member of B_0.

Case 2. $B_0 \notin F^{h_n^1}$ for any $n \in \omega$.

Thus $B_0 \in F^{h^2}$. Without loss of generality, we can assume the i_3 associated with g_3 is less than or equal to γ^3.

Let $i_4 = \mathrm{MAX}(i_3, \mathrm{TOP}(g_1))$.

Let $i_5 = \mathrm{CEIL}(A^{h^2} \restriction (i_4 \times \omega))$.

Subcase 2.1. $i_4 = \mathrm{TOP}(g_1)$.

Pick n_0 so that $g_0 \restriction (i_5 \times \omega) \in A_{i_5}^{h_{n_0}^1}$. There exists $g_1^1 \in A_{i_5}^{h_{n_0}^1}$ such that $g_1^1 \supseteq g_0 \restriction (i_5 \times \omega)$ and g_1^1 is above some member of $(B_0)_{i_5}^{g_2}$. There exists $g_2^1 \in A^{h^2}$ such that $g_2^1 \supseteq g_2$ and g_2^1 is above some member of B_0 and $g_2^1 \restriction (i_4 \times \omega) \subseteq g_1^1$. Then $(g_0 \cup g_1^1 \cup g_2^1) \restriction (i \times \omega) \in A_i^{h^2}$ and $(g_0 \cup g_1^1 \cup g_2^1) \restriction (i \times \omega)$ is above some member of B_0.

Subcase 2.2. $i_4 = i_3$.

Pick n_0 so that:

(1) $g^3 \restriction (\mathrm{DOM}(g^3) \setminus c_{n_0}) \cap \mathrm{FI}(A^{h^2}) \times \omega = g \restriction (\mathrm{DOM}(g) \setminus c_{n_0}) \cap \mathrm{FI}(A^{h^2}) \times \omega$.

(2) $g \supseteq p_{n_0} \restriction (\mathrm{DOM}(p_{n_0}) \setminus c_{n_0}) \cap (\gamma^3 \times \omega)$.

(3) If $g' = p_{n_0}$ changed on a subset of c_{n_0}, then p' is above some member of $(B_0)^{g_2}_{i_5}$.

Let g_3^1 be such that $\mathrm{DOM}(g_3^1) = \mathrm{DOM}(g_3) \cup [c_{n_0} \cap (i_5 \times \omega)]$ and $g_3^1 \supseteq g_3$ and $g_0 \cup g_3^1$ is still a function. Thus, g_3^1 is above some member of $(B_0)^{g_2}_{i_5}$. There exists $g_2^1 \in A^{h_2}$ such that g_2^1 is above some member of B_0 and $g_2^1 \supseteq g_2$ and $g_3^1 \supseteq g_2^1 \restriction (i_5 \times \omega)$. Thus, $(g_0 \cup g_2^1 \cup g_3^1) \restriction (i \times \omega) \in A_i^{h_1^2}$ and $(g_0 \cup g_2^1 \cup g_3^1) \restriction (i \times \omega)$ is above some member of B_0. Thus, h_1^2 satisfies B-maximality.

We now check that 1 (in def of α-candidate), i.e., closure under finite changes, holds. Clearly, g_1, g_2, g_3 are all "closed under finite deletions" in the sense that if we delete one point from the domain of g_i ($i = 1,2,3$), we still have a valid g_i. Finite additions follow from one addition by induction and a "change" is a deletion followed by an addition. So suppose we wish to add β, n_0 to dom(g), where $g = g_1 \cup g_2 \cup g_3$. Thus, $\beta + 1 \in \mathrm{DOM}(h_2)$ or $\beta + 1 \in \mathrm{DOM}(h_1^n)$ for some n. In the first case, β, n_0 can be added to the domain of g_2. In the second case, β, n_0 can be added to the domain of g_1.

Note that hypothesis 10 guarantees that requirement 2 in the definition of α-candidate holds. □

V. Shelah forcing

We now define the partial orders we will force with.

5.1. Definition. For $\alpha < \omega_2$:

Let $\mathbf{P}_\alpha = \{h : h$ is an α-candidate and

$$\forall i < \alpha \ \forall g \in A^h [h \restriction (i+1) \Vdash^{\mathbf{P}_i} \omega \setminus \mathrm{DOM}(f^i_g) \in \tau_i]\}$$

where τ_i is a name in the forcing language for $\mathbf{P}_i$ and $0 \Vdash^{\mathbf{P}_i}$ "τ_i is a P-filter". Let $\mathbf{P}_{\omega_2} = \bigcup_{\alpha < \omega_2} \mathbf{P}_\alpha$. ($f^i_g$ was defined in III.)

Notice that $\alpha < \beta \Rightarrow \mathbf{P}_\alpha \subseteq \mathbf{P}_\beta$. Our next proposition shows that iterated forcing works for the $\mathbf{P}_\alpha$'s. Also notice that $\alpha < \beta$ and $h \in \mathbf{P}_\beta \Rightarrow h \restriction (\alpha + 1) \in \mathbf{P}_\alpha$. $\mathbf{P}_\alpha$ depends on the sequence τ_i ($i < \alpha$).

The gluing lemma deals with α-candidates. The glued together α-candidate h_1^2 in the conclusion of the gluing lemma is bigger than the other candidates in the hypothesis of the gluing lemma. If all candidates in the hypothesis of the gluing lemma are in $\mathbf{P}_\alpha$ and

$$h_1^2 \restriction (i+1) \Vdash^{P_i} \omega \setminus \mathrm{DOM}(f_g^i) \in \tau_i \qquad \text{for all } i < \alpha$$

where g is the function over the covering sequence, then $h_1^2 \in \mathbf{P}_\alpha$. This follows from the fact that τ_i is a filter. We will use the fact, without mentioning it, that the h_1^2 is actually in $\mathbf{P}_\alpha$ and not merely an α-candidate.

5.2. PROPOSITION. Hypothesis:

(1) $\alpha < \beta \leqq \omega_2$,

(2) $h_1 \in \mathbf{P}_\alpha$, $h_2 \in \mathbf{P}_\beta$,

(3) $h_1 \geqq h_2 \restriction (\alpha + 1)$.

Conclusion:

There exists an $h_3 \in \mathbf{P}_\beta$ such that $h_3 \geqq h_1, h_2$.

PROOF. Follows from the gluing lemma, where we take $\gamma^2 = \beta$, $\gamma_n^1 = \alpha$, $\gamma^3 = 0$, $g = \varnothing$, and the sequence h_n^1 to be the constant sequence h_1. □

This proposition guarantees us that every generic set on $\mathbf{P}_\beta$ "restricted" to $\mathbf{P}_\alpha$ is $\mathbf{P}_\alpha$-generic. Every generic set on $\mathbf{P}_\alpha$ is extendable to one for $\mathbf{P}_\beta$. Furthermore if $\tilde{\sigma}$ is a $\mathbf{P}_\alpha$-name and $G_\alpha \subseteq G_\beta$ are $\mathbf{P}_\alpha$, $\mathbf{P}_\beta$ generic respectively then $\tilde{\sigma}$ names the same set in $M[G_\alpha]$ and $M[G_\beta]$. For more details, see chapters 7 and 8 of [4].

Our next proposition explains why we chose to define $\mathbf{P}_\alpha$ the way we did.

5.3. PROPOSITION. *For all $h \in \mathbf{P}_\alpha$, $h \Vdash^{\mathbf{P}_\alpha} A^h \subseteq Q_{\langle \tau_j : j < \alpha \rangle}$.*

PROOF. Follows from 5.1. □

Our next proposition shows us that $\mathbf{P}_\alpha$ is $\aleph_1$-complete.

5.4. PROPOSITION. *If h_n $(n \in \omega)$ is an increasing sequence of members of $\mathbf{P}_\alpha$, then there exists an $h \in \mathbf{P}_\alpha$ such that $h \geqq h_n$ for all $n \in \omega$ and $A^h = \bigcup_{n \in \omega} A^{h_n}$.*

PROOF. Follows from the gluing lemma, taking $\gamma^2 = \alpha$, $h^2 \in \mathbf{P}_0$, $\gamma_n^1 = \alpha$, and $\gamma^3 = 0$. □

It is always possible to assume GCH in the ground model and we do so here. For more details on Δ-system arguments, see [4], especially theorem 1.6 of chapter 2.

5.5. LEMMA. *For all $\alpha \leqq \omega_2$, $\mathbf{P}_\alpha$ has the ω_2-cc.*

PROOF. Since we assume the GCH in the ground model, $|\mathbf{P}_\alpha| = \aleph_1$ for $\alpha < \omega_2$. The lemma follows by a Δ-system argument from

(*) Assume $h_1 \in \mathbf{P}_\alpha$, $h_2 \in \mathbf{P}_\beta$, $\alpha < \beta$, and that there exists a $\gamma \leqq \alpha$ such that

$h_2 \restriction (\gamma+1) = h_2 \restriction (\alpha+1)$ and $h_1 \restriction (\gamma+1)$, $h_2 \restriction (\gamma+1)$ are compatible in $\mathbf{P}_\gamma$. Then h_1, h_2 are compatible in $\mathbf{P}_\beta$.

Proof of (*). Let $h_3 \in \mathbf{P}_\gamma$ be such that $h_3 \geqq h_2 \restriction (\gamma+1)$, $h_1 \restriction (\gamma+1)$. By Proposition 5.2, there exists an $h_4 \in \mathbf{P}_\alpha$ such that $h_4 \geqq h_3, h_1$. By Proposition 5.2, there exists an $h_5 \in \mathbf{P}_\beta$ such that $h_5 \geqq h_2, h_4$. Thus, $h_5 \geqq h_1, h_2$. This completes the proof of (*). □

The next lemma tells us that for every covering sequence for h, we can extend h to include a g which is over the given covering sequence.

5.6. Lemma. *For every covering sequence* $\overrightarrow{\langle p_n, c_n \rangle}$ *for* $h \in \mathbf{P}_\beta$, *and for every* $n_0 \in \omega$ *there exists an infinite* $S \subseteq \omega$ *and an* $h' \in \mathbf{P}_\beta$ *and a* $g \in A^{h'}$ *such that*:

(1) $n_0 \in S$,

(2) $g \supseteq p_{n_0}$,

(3) $h' \geqq h$,

(4) g *is over* $\overrightarrow{\langle p_n, c_n \rangle}$ *on* S *for* β.

Proof. Follows from the next proposition by taking $\alpha = 0$, $S_0 = \omega$, $h_3 = h'$, and $g_3 = g$. We will not need the next proposition again. □

5.7. Proposition. Hypothesis:

(1) $h_1 \in \mathbf{P}_\alpha$, $h_2 \in \mathbf{P}_\beta$, *and* $\alpha < \beta$,

(2) $h_1 \geqq h_2 \restriction (\alpha+1)$,

(3) $\overrightarrow{\langle p_n, c_n \rangle}$ *is a covering sequence for* h_2,

(4) $g_1 \in A^{h_1}$ *and* g_1 *is over* $\overrightarrow{\langle p_n, c_n \rangle}$ *on* S_0 *for* α.

Conclusion:

For each $n_0 \in S_0$ *there exists an infinite* $S_1 \subseteq S_0$ *and an* $h_3 \in \mathbf{P}_\beta$ *and a* $g_3 \in A^{h_3}$ *such that*

(1) $S_1 \cap [0, n_0] = S_0 \cap [0, n_0]$,

(2) $g_3 \restriction (\alpha \times \omega) = g_1$,

(3) $g_3 \restriction (\beta \setminus \alpha) \times \omega \supseteq p_{n_0} \restriction (\beta \setminus \alpha) \times \omega$,

(4) g_3 *is over* $\overrightarrow{\langle p_n, c_n \rangle}$ *on* S_1 *for* β,

(5) $h_3 \geqq h_1, h_2$.

Proof. We prove the proposition by induction on $\beta \leqq \omega_2$. If $\mathrm{CF}(\beta) \geqq \omega_1$, there is no problem.

Case 1. β is a successor.

By the induction hypothesis, we can assume, without loss of generality, that $\beta = \alpha + 1$. If $\beta \notin \mathrm{DOM}(h_2)$, then take $g_3 = g_1$, $S_1 = S_0$, and $h_3 = h_1$. So assume

$\beta \in \mathrm{DOM}(h_2)$. Let $c_n^\alpha = \{i \in \omega : \langle \alpha, i\rangle \in c_n\}$. Thus, $\bigcup_{n\in\omega} c_n^\alpha = \omega$. We can apply the collecting lemma to get an infinite $S_1 \subseteq S_0$, an $E \subseteq \omega$, and an $h_1^1 \in \mathbf{P}_\alpha$ such that $S_1 \cap [0, n_0] = S_0 \cap [0, n_0]$, $h_1^1 \geqq h_1$, $h_1^1 \Vdash^{\mathbf{P}_\alpha} E \in \mathcal{I}_{\tau_\alpha}$, $E \supseteq \mathrm{DOM}(f^\alpha_{p_{n_0}})$, and $\forall n \in S_1[E \supseteq \mathrm{DOM}(f^\alpha_{p_n}) \setminus c_n^\alpha]$. Define g_3 so that $g_3 \restriction (\alpha \times \omega) = g_1$ and $f^\alpha_{g_3} = \bigcup_{n\in\omega} (f^\alpha_{p_n}) \restriction E$. The result follows from the gluing lemma by taking $\gamma_n^1 = \alpha$, $\gamma^2 = \gamma^3 = \beta$, $g = g_3$, $h_n^1\ (n \in \omega)$ to be the constant sequence h_1^1, and h^2 to be h_2.

Case 2. $\mathrm{CF}(\beta) = \omega$.

Let $\alpha_n\ (n \in \omega) \nearrow \beta$ and $\alpha_0 = \alpha$. By the induction hypothesis, we can assume the lemma holds if β is replaced by α_n. Let $f(0) = n_0$, $g^0 = g_1$, $h^0 = h_1$, and $S^0 = S_0$. By induction on n, we define $h^n \in \mathbf{P}_{\alpha_n}$, $S^n \subseteq \omega$, $g^n \in A^{h^n}$, $f(n) \in \omega$ to satisfy:

(1) $f(n) > f(n-1)$ and $f(n) \in S_{n-1}$,

(2) $S_n \subseteq S_{n-1}$, S_n infinite, and $S_n \cap [0, f(n)] = S_{n-1} \cap [0, f(n)]$,

(3) $g^n \restriction (\alpha_{n-1} \times \omega) = g^{n-1}$,

(4) $g^n \restriction (\alpha_n \setminus \alpha_{n-1}) \times \omega \supseteq p_{f(n)} \restriction (\alpha_n \setminus \alpha_{n-1}) \times \omega$,

(5) g^n is over $\overrightarrow{\langle p_n, c_n\rangle}$ on S_n for α_n,

(6) $h^n \geqq h^{n-1}$ and $h^n \geqq h_2 \restriction (\alpha_n + 1)$.

This can be done by the inductive hypothesis. Define $g_3 = \bigcup_{n\in\omega} g^n$. Let $S_1 = \bigcap_{n\in\omega} S_n$. Clearly, (1), (2), (3) in the conclusion of the lemma hold. Since $S_1 \supseteq \mathrm{image}(f)$, S_1 is infinite. Since $g_3 \restriction (\alpha_n \times \omega)$ is over $\overrightarrow{\langle p_n, c_n\rangle}$ on S^n for α_n, $S_1 \subseteq S^n$, and $\alpha_n \nearrow \beta$, we get that (4) in the conclusion of the lemma holds. The lemma follows from the gluing lemma, where we take γ_n^1 to be α_n, $\gamma^1 = \gamma^2 = \gamma^3 = \beta$, $h_n^1\ (n \in \omega)$ to be h^n, and h^2 to be h_2. The h_1^2 in the conclusion of the gluing lemma we take to be our desired h_3. □

5.8. Proposition. *For any $h \in \mathbf{P}_\beta$ and any countable $C \subseteq \beta + 1$, it is possible to find an $h' \in \mathbf{P}_\beta$ such that $h' \geqq h$ and* $\mathrm{DOM}(h') \supseteq C$.

Proof. Since $\mathbf{P}_\beta$ is $\aleph_1$-complete, it is sufficient if we consider the case where C has only one element. So assume $C = \{\alpha\}$. If $\alpha \in \mathrm{DOM}(h)$, let $h' = h$. So assume $\alpha \notin \mathrm{DOM}(h)$. If α is a limit, let $\alpha_1 = \sup(\alpha \cap \mathrm{DOM}(h))$. Define $h'(i) = h(i)$ for $i \neq \alpha$, and $A_\alpha^{h'} = A_{\alpha_1}^h$, $F_\alpha^{h'} = F_{\alpha_1}^h$. This takes care of the limit case, so we can assume that $\alpha = \alpha_1 + 1$. Define h_2 so that $\mathrm{DOM}(h_2) = \{0, \alpha\}$ and

$$A_\alpha^{h_2} = \{f : f : \{\alpha_1\} \times \omega \xrightarrow{\text{partial}} \{0,1\} \text{ and } \mathrm{DOM}(f) \text{ is finite}\}.$$

Let $F_\alpha^{h_2} = \{B_\alpha^f : f \in A_\alpha^{h_2}\} \cup \{\{\varnothing\}\}$. Hence $h_2 \in \mathbf{P}_\alpha$. Since $\mathrm{DOM}(h_2 \restriction \alpha) = \{0\}$, $h_2 \restriction \alpha \leqq h \restriction \alpha$. By 5.2, there exists an $h_3 \in \mathbf{P}_\alpha$ such that $h_3 \geqq h_2, h \restriction (\alpha + 1)$. By 5.2, there exists an $h' \in \mathbf{P}_\beta$ such that $h' \geqq h, h_3$. Since $h' \geqq h_2$, $\mathrm{DOM}(h') \supseteq \{\alpha\}$. □

5.9. Remark. Because $\mathbf{P}_{\omega_2}$ is ω_1-complete and ω_2-cc, there are only ω_2

P-filters and any one in $V^{\mathbf{P}_{\omega_2}}$ is in $V^{\mathbf{P}_\alpha}$ for some $\alpha < \omega_2$. Thus for any P-filter $\mathcal{D}$ in $V^{\mathbf{P}_{\omega_2}}$, we can arrange that $\mathcal{D}$ is named by τ_i for some $i < \omega_2$ and that $\tau_i = \tau_{i+n}$ for all $n \in \omega$. We will assume this from now on.

From now on, when we write $\mathbf{P}$, we mean $\mathbf{P}_{\omega_2}$.

5.10. Definition. In $V^{\mathbf{P}}$, define $\mathbf{Q} = \bigcup\{A^h : h \in G\}$ where G is the generic set for $\mathbf{P}$. The partial order on $\mathbf{Q}$ is: $g_1 \leqq g_2$ iff $g_1 \subseteq g_2$.

5.11. Remark. Our goal is to prove that there are no P-points in $(V^{\mathbf{P}})^{\mathbf{Q}}$. We will do this by showing no P-filter in $V^{\mathbf{P}}$ can be extended to a P-point in $(V^{\mathbf{P}})^{\mathbf{Q}}$ and then use Lemma 2.1. In order to apply 2.1, we need to know that $\mathbf{Q}$ has the ccc. The whole purpose of the B's in Shelah forcing is to guarantee that $\mathbf{Q}$ will have the ccc. Our next lemma verifies this.

5.12. Lemma. *In $V^{\mathbf{P}}$, $\mathbf{Q}$ has the ccc.*

Proof. Suppose not. There is an $h \in \mathbf{P}$ such that

$$h \Vdash^{\mathbf{P}} \text{“}\tilde{\sigma} \text{ is a maximal uncountable antichain in } \mathbf{Q}\text{”}.$$

If $g \in A^h$, then $h \Vdash^{\mathbf{P}} g \in \mathbf{Q}$. Thus, there exists an $h' \in \mathbf{P}$ and a $g' \in A^{h'}$ such that $h' \geqq h$, $g' \supseteq g$, and $h' \Vdash^{\mathbf{P}}$ “g' is above some member of $\tilde{\sigma}$”. By repeating this construction once for each $g \in A^h$ and applying 5.4 we get:

(*) There exists an $h_1 \geqq h$, $h_1 \in \mathbf{P}$ such that for all $g \in A^h$ there exists a $g_1 \in A^{h_1}$ such that $g_1 \supseteq g$ and $h_1 \Vdash^{\mathbf{P}}$ “g_1 is above some member of $\tilde{\sigma}$”.

By repeating (*) ω times applying 5.4, we get an $h_2 \geqq h$, $h_2 \in \mathbf{P}$ such that for all $g \in A^{h_2}$ there exists a $g' \in A^{h_2}$ such that $g' \supseteq g$ and $h_2 \Vdash^{\mathbf{P}}$ “g' is above some member of $\tilde{\sigma}$”. Let $B = \{g \in A^{h_2} : h_2 \Vdash^{\mathbf{P}}$ “g is above some member of $\tilde{\sigma}$”$\}$. Define h_3 so that $\mathrm{DOM}(h_3) = \mathrm{DOM}(h_2)$ and for all $i \in \mathrm{DOM}(h_3)$, let $A_i^{h_3} = A_i^{h_2}$ and $F_i^{h_3} = F_i^{h_2} \cup \{B \restriction i \times \omega\}$. Thus, h_3 is an $\aleph_2$-candidate (except for B-bounding). By 4.6, there exists an $h_4 \in \mathbf{P}$ such that $h_4 \geqq h_2$ and $B \in F^{h_4}$. Since $B \in F^{h_4}$ we have that $h_4 \Vdash^{\mathbf{P}}$ “every member of $\tilde{\sigma}$ can be extended to be above a member of B”. Also, we have that $h_4 \Vdash^{\mathbf{P}}$ “every member of B is above some member of $\tilde{\sigma}$; B is countable; $\tilde{\sigma}$ is uncountable”. Putting these together, we get that $h_4 \Vdash^{\mathbf{P}}$ “$0 = 1$”. Contradiction. Thus, $\mathbf{Q}$ has the ccc. □

VI. Conclusion of the proof

The next lemma will be useful in proving that every “new” element of ω^ω is bounded by an old one. Our next lemma says if we start with a name for a function with domain ω, for an infinite subset of the domain we can find a finite set of possible values for each member of the infinite subset of the domain.

6.1. LEMMA. *For each $h \in \mathbf{P}$ such that $h \Vdash^{\mathbf{P}}$ "$g \in \mathbf{Q}$ and $\varnothing \Vdash^{\mathbf{Q}} \tilde{\tau}: \omega \to V$" there exists an infinite $S \in 2^{\omega} \cap V$ and a sequence T_n $(n \in S)$ in V and an $h' \in \mathbf{P}$ and a $g' \in A^{h'}$ such that*:

(1) $g' \supseteq g$ *and* $h' \geqq h$,

(2) $|T_n| \leqq n$,

(3) $h' \Vdash^{\mathbf{P}}$ "$g' \Vdash^{\mathbf{Q}} \forall n \in S(\tilde{\tau}(n) \in T_n)$".

PROOF. By extending h if necessary, we can assume without loss of generality, that $g \in A^h$.

(*) For each $g_0 \in A^h$ and $n \in \omega$, we can find an $h_1 \geqq h$ and a $g_1 \in A^{h_1}$ and a $t \in V$ such that $g_1 \supseteq g_0$ and $h_1 \Vdash^{\mathbf{P}}$ "$g_1 \Vdash^{\mathbf{Q}} \tilde{\tau}(n) = t$".

By applying (*), repeatedly, we can find an $h_2 \in \mathbf{P}$ such that $h_2 \geqq h$ and for all $g_2 \in A^{h_2}$ and all $n \in \omega$ there exists a $t \in V$ and a $g_2' \in A^{h_2}$ such that $g_2' \supseteq g_2$ and $h_2 \Vdash^{\mathbf{P}}$ "$g_2' \Vdash^{\mathbf{Q}} \tilde{\tau}(n) = t$".

Let c_n be an increasing sequence of finite sets such that $|c_n| \leqq \log_2(n)$ and $\bigcup_{n \in \omega} c_n = \mathrm{FI}(A^{h_2}) \times \omega$. Let $B_1, B_2, \cdots$ be a list of all the elements of F^{h_2}. Let $p_0 = g$. We now define p_n $(n \geqq 1)$ inductively to satisfy:

(a) $p_n \geqq p_{n-1}$ and $p_n \in A^{h_2}$.

(b) $\mathrm{DOM}(p_n) \supseteq c_n$.

(c) If p' is p_n changed on a subset of c_n, then it is above some member of B_i $(i \leqq n)$.

(d) $|T_n| \leqq n$ and if p' is p_n changed on a subset of c_n, then

$$h_2 \overset{\mathbf{P}}{\Vdash} [p' \overset{\mathbf{Q}}{\Vdash} \tilde{\tau}(n) \in T_n].$$

Assume p_{n-1} is defined. We show how p_n and T_n can be defined. First extend p_{n-1} to p_n^0 so that (a), (b), (c) hold, by 4.10. Let $\{s_1, \cdots, s_k\} = 2^{c_n}$. Define p_n^i $(1 \leqq i \leqq k)$, t^i inductively to satisfy:

(i) $p_n^i \supseteq p_n^{i-1}$ and $p_n^i \in A^{h_2}$,

(ii) $h_2 \Vdash^{\mathbf{P}} [p_n^i \restriction [\mathrm{DOM}(p_n^i) \setminus c_n] \cup s_i \Vdash^{\mathbf{Q}} \tilde{\tau}(n) = t_i]$.

Let $p_n = p_n^k$. Let $T_n = \{t_1, \cdots, t_k\}$. $|T_n| \leqq k = 2^{|c_n|} \leqq 2^{\log_2 n} = n$. Thus (a), (b), (c), (d) hold. By 5.6 there exists an $h' \geqq h_2$ and a $g' \in A^{h'}$ and an infinite $S \subseteq \omega$ such that g' is over $\overrightarrow{\langle p_n, c_n \rangle}$ on S for ω_2, $g' \supseteq p_0 = g$, and $0 \not\in S$. Therefore, (d) holds for $n \in S$. Since $g' \supseteq p_n \restriction [\mathrm{DOM}(p_n) \setminus c_n]$ for all $n \in S$, we have that (3) in the conclusion holds. □

The next lemma will tell us that every "new" $f \in \omega^\omega$ is bounded by an "old" one.

6.2. LEMMA. *If $h \Vdash^{\mathbf{P}} [g \in \mathbf{Q}$ and $\varnothing \Vdash^{\mathbf{Q}} \tilde{\tau} : \omega \to \omega]$, then there exists an $f_1 \in V$, $h_1 \in \mathbf{P}$, and $g_1 \in A^{h_1}$ such that $h_1 \geqq h$, $g_1 \supseteq g$, and*

$$h_1 \overset{\mathbf{P}}{\Vdash} (g_1 \overset{\mathbf{Q}}{\Vdash} \forall n \in \omega[\tilde{\tau}(n) \leqq f_1(n)]).$$

PROOF. Without loss of generality, we can assume that $\tilde{\tau}$ names an increasing function. By 6.1, we can assume there is an $h_1 \geqq h$, a $g_1 \supseteq g$, an infinite $S \subseteq \omega$, and a sequence T_n $(n \in \omega)$ in V such that $|T_n| \leqq n$ and

$$h_1 \overset{\mathbf{P}}{\Vdash} [g_1 \overset{\mathbf{Q}}{\Vdash} \forall n \in S(\tilde{\tau}(n) \in T_n)].$$

Let $f_0(n) =$ least element of S greater than or equal to n. Define $f_1(n) = \max(T_{f_0(n)})$. Thus,

$$h_1 \overset{\mathbf{P}}{\Vdash} [g_1 \overset{\mathbf{Q}}{\Vdash} \forall n \in S(\tilde{\tau}(n) \leqq f_1(n))].$$

For all $n \in \omega$ there exists a least $m \in S$ such that $m \geqq n$. Thus $f_0(n) = f_0(m)$ for this $m \in S$. Hence, $f_1(n) = f_1(m)$ for this $m \in S$. Therefore, since τ is increasing,

$$h_1 \overset{\mathbf{P}}{\Vdash} [g_1 \overset{\mathbf{Q}}{\Vdash} \forall n \in \omega(\tilde{\tau}(n) \leqq f_1(n))]. \qquad \square$$

6.3. LEMMA. *For each $f \in (V^{\mathbf{P}})^{\mathbf{Q}}$ such that $f : \omega \to \omega$ there exists an $f_1 \in V$ such that $f_1 : \omega \to \omega$ and $\forall n \in \omega[f(n) \leqq f_1(n)]$.*

PROOF. Follows from 6.2. $\square$

6.4. LEMMA. *No P-filter $\mathcal{D}$ in $V^{\mathbf{P}}$ can be extended to a P-point $\mathcal{D}_0$ in $(V^{\mathbf{P}})^{\mathbf{Q}}$.*

PROOF. Since $\mathcal{D}$ is a P-filter in $V^{\mathbf{P}}$ it is named by τ_{α_0+n} $(n \in \omega)$ for some $\alpha_0 \in \aleph_2$. In $(V^{\mathbf{P}})^{\mathbf{Q}}$, we let $f_i = \bigcup\{f^i_g : g$ is in the generic set for $\mathbf{Q}\}$ and we let $A_i = \{l \in \omega : f_i(l) = 1\}$. In $(V^{\mathbf{P}})^{\mathbf{Q}}$, let $\varepsilon : \omega \to 2$ be such that $A^{\varepsilon(n)}_{\alpha_0+n} \in \mathcal{D}_0$ for all $n \in \omega$, where $A^0 = A$ and $A^1 = \omega \setminus A$.

In $(V^{\mathbf{P}})^{\mathbf{Q}}$, let $\tau : \omega \to 2^{<\omega}$ be given by $\tau(n) = \varepsilon \restriction (n+3)2^n$. In $V^{\mathbf{P}}$, we get a $g_2 \in \mathbf{Q}$ and an infinite $S \in 2^\omega \cap V$ and a sequence T_n $(n \in S)$ in V such that $|T_n| \leqq n$ $(n \in S)$ and

$$g_2 \overset{\mathbf{Q}}{\Vdash} \forall n \in S[\tau(n) \in T_n] \;\&\; \forall n \in \omega[A \setminus A^{\varepsilon(n)}_{\alpha_0+n} \subseteq [0, f^1(n))].$$

We also, of course, require that $g_2 \Vdash^{\mathbf{Q}}$ "$\mathcal{D} \subseteq \tilde{\mathcal{D}}_0$".

We now prove that there exist sequences i_n, j_n $(n \in S)$ in $V^{\mathbf{P}}$ (hence in V) such that $i_n, j_n > n$, $i_n \neq j_n$, and $g_2 \Vdash^{\mathbf{Q}} \varepsilon(i_n) = \varepsilon(j_n)$. For $n \in S$, let $T_n = \{\varepsilon_1, \cdots, \varepsilon_n\}$. Let $K_0 = \{j : 0 \leqq j < (n+3)2^n\}$. Pick $K_1 \subseteq K_0$ so that for all $i, j \in K_1$, $\varepsilon_1(i) = \varepsilon_1(j)$ and $|K_1| \geqq \frac{1}{2} \cdot |K_0|$. Continue this procedure n times getting $K_0 \supseteq K_1 \supseteq \cdots \supseteq K_n$ and $|K_n| \geqq 2^{-n} \cdot |K_0|$. We can pick $i_n, j_n > n$ with $i_n \neq j_n$ and $i_n, j_n \in K_n$ since $|K_n| \geqq (n+3)$.

By taking a subsequence, we can assume that we have sequences i_n $(n \in \omega)$, j_n $(n \in \omega)$ in V such that $i_{n+1} > i_n, j_n$ and $j_{n+1} > i_n, j_n$ and $i_n \neq j_n$ and in $V^{\mathbf{P}}$, we have that $g_2 \Vdash^{\mathbf{Q}} \forall n \in \omega[\varepsilon(i_n) = \varepsilon(j_n)]$.

Let $\tilde{E}_n$ be a name (in the forcing language for $\mathbf{Q}$) for $[A_{\alpha_0+i_n} \cap A_{\alpha_0+j_n}] \cup [(\omega \setminus A_{\alpha_0+i_n}) \cap (\omega \setminus A_{\alpha_0+j_n})]$. Thus, we have in $V^{\mathbf{P}}$ that $g_2 \Vdash^{\mathbf{Q}} \forall n \in \omega[\tilde{E}_n \in \mathscr{D}_0]$. In $V^{\mathbf{P}}$, we can find a $g_3 \in \mathbf{Q}$, $\tilde{\sigma}$ a name in the forcing language for $\mathbf{Q}$ of a subset of ω, and an increasing $f^2 \in V \cap \omega^\omega$ such that $g_3 \supseteq g_2$ and $g_3 \Vdash^{\mathbf{Q}} \tilde{\sigma} \in \mathscr{D}_0$ & $\forall n \in \omega[\tilde{\sigma} \setminus \tilde{E}_n \subseteq [0, f^2(n))]$.

Since we have been working mostly in $V^{\mathbf{P}}$, we can take an $h_3 \in \mathbf{P}$ to force everything we have discussed. To review, h_3 forces the following for some $g_3 \in A^{h_3}$:

(2) $g_3 \Vdash^{\mathbf{Q}} \varepsilon(i_n) = \varepsilon(j_n)$ for all $n \in \omega$,

(3) $g_3 \Vdash^{\mathbf{Q}} \forall n \in \omega[\tilde{E}_n \in \mathscr{D}_0]$,

(4) $g_3 \Vdash^{\mathbf{Q}} \tilde{\sigma} \in \mathscr{D}_0$ & $\tilde{\sigma} \setminus \tilde{E}_n \subseteq [0, f^2(n))$ where $f^2 \in V \cap \omega^\omega$.

Without loss of generality, we can assume that $\mathrm{DOM}(h_3) \supseteq \{\alpha_0, \alpha_0 + 1, \cdots, \alpha_0 + \omega\}$.

Let $h_4 = h_3 \restriction \alpha_0 + \omega + 1$ and $g_4 = g_3 \restriction (\alpha_0 + \omega) \times \omega$. Let $\overrightarrow{\langle p_n, c_n \rangle}$ be a covering sequence for h_4 with $g_4 \subseteq p_0$. By 5.6, we can find an $h_5 \in \mathbf{P}_{\alpha_0+\omega}$ and a $g_5 \in A^{h_5}$ and an infinite $S_1 \in V \cap 2^\omega$ such that g_5 is over $\overrightarrow{\langle p_n, c_n \rangle}$ on S_1 for $\alpha_0 + \omega$ and $g_5 \supseteq p_0$, and $h_5 \geqq h_4$. We let $Y_n^{(1)} = \mathrm{DOM}(f_{g_5}^{\alpha_0+n})$ for all $n \in \omega$. Hence, we have that

$$h_5 \restriction (\alpha_0 + 1) \Vdash^{\mathbf{P}_{\alpha_0}} Y_n^{(1)} \in \mathscr{I}_{\tau_{\alpha_0}}.$$

By the definition of P-filter and ω_1-completeness, find $h_6 \in \mathbf{P}_{\alpha_0}$ with $h_6 \geqq h_5 \restriction (\alpha_0 + 1)$ and an increasing $f^3 \in V \cap \omega^\omega$ and a $Y_0 \in V \cap 2^\omega$ such that $h_6 \Vdash^{\mathbf{P}_{\alpha_0}} Y_0 \in \mathscr{I}_{\tau_{\alpha_0}}$ and $Y_0 \cup [0, f^{(3)}(n)) \supseteq Y_n^{(1)}$ for all $n \in \omega$. Pick $g_6 : (\alpha_0 + \omega) \times \omega \xrightarrow{\text{partial}} \{0, 1\}$ such that $g_6 \supseteq g_5$, $g_6 \restriction (\alpha_0 \times \omega) = g_5 \restriction (\alpha_0 \times \omega)$ and $\forall n \in \omega[\mathrm{DOM}(f_{g_6}^{\alpha_0+n}) = Y_0 \cup [0, f^{(3)}(n))]$. Since $g_6 \supseteq g_5$, we have that g_6 is over $\overrightarrow{\langle p_n, c_n \rangle}$ on S_1 for $\alpha_0 + \omega$. Pick an increasing $f^4 \in V \cap \omega^\omega$ such that

$$f^4(n) \geqq \max\{f^3(i_n), f^3(j_n), f^2(n), f^1(n)\}.$$

Let $k \notin Y_0$ and $f^{(4)}(n+1) > k \geqq f^4(n)$. In this case, notice that $\langle \alpha_0 + i_n, k \rangle \notin \mathrm{DOM}(g_6)$ and $\langle \alpha_0 + j_n, k \rangle \notin \mathrm{DOM}(g_6)$. Define g_7 so that $\mathrm{DOM}(g_7) =$

$\mathrm{DOM}(g_6) \cup \{\langle \alpha_0 + i_n, k\rangle : n \in \omega, \quad k \notin Y_0, \quad \text{and} \quad f^4(n+1) > k \geqq f^4(n)\} \cup \{\langle \alpha_0 + j_n, k\rangle : n \in \omega,\ k \notin Y_0, \text{ and } f^4(n+1) > k \geqq f^4(n)\}$, and so that:

(5) $g_7 \supseteq g_6$,

(6) $g_7(\alpha_0 + i_n, k) = 0$ for $n \in \omega$, $k \notin Y_0$, and $f^4(n+1) > k \geqq f^4(n)$,

(7) $g_7(\alpha_0 + j_n, k) = 1$ for $n \in \omega$, $k \notin Y_0$, and $f^4(n+1) > k \geqq f^4(n)$.

We now apply the gluing lemma with $\gamma_n^1 = \alpha_0$ and h_n^1 $(n \in \omega)$ the constant sequence h_6. $\gamma^2 = \gamma^3 = \alpha_0 + \omega$, $h^2 = h_4$, and $g = g_7$. This gives us an $h_7 \in \mathbf{P}_{\alpha_0+\omega}$ such that $g_7 \in A^{h_7}$ and $h_7 \geqq h_6, h_4$. By 5.2, we can find an $h_8 \in \mathbf{P}$ such that $h_8 \geqq h_7$, h_3. We let $g_8 = g_7 \cup g_3$. Thus, $g_8 \in A^{h_8}$.

We will now show that $h_8 \Vdash^{\mathbf{P}}$ "$0 = 1$" which will finish the proof of the lemma. Assume that $k \in [f^4(n_0), f^4(n_0+1))$ for some $n_0 \geqq 1$, and that $k \notin Y_0$. Since

$$h_8 \Vdash^{\mathbf{P}} [g_8 \Vdash^{\mathbf{Q}} \tilde{\sigma} \setminus \tilde{E}_n \subseteq [0, f^{(4)}(n)) \text{ for all } n \in \omega],$$

we have that

$$h_8 \Vdash^{\mathbf{P}} [g_8 \Vdash^{\mathbf{Q}} k \notin \tilde{\sigma} \setminus \tilde{E}_{n_0}].$$

Since $g_8(\alpha_0 + i_{n_0}, k) = 0$ and $g_8(\alpha_0 + j_{n_0}, k) = 1$, we have

$$h_8 \Vdash^{\mathbf{P}} [g_8 \Vdash^{\mathbf{Q}} k \notin \tilde{E}_{n_0}].$$

Thus, $h_8 \Vdash^{\mathbf{P}} [g_8 \Vdash^{\mathbf{Q}} k \notin \tilde{\sigma}]$. Since $n_0 \geqq 1$ was arbitrary, we have that

$$h_8 \Vdash^{\mathbf{P}} [g_8 \Vdash^{\mathbf{Q}} \tilde{\sigma} \subseteq Y_0 \cup [0, f^4(1))].$$

Since $h_8 \Vdash^{\mathbf{P}} \omega \setminus Y_0 \in \tau_{\alpha_0}$, we have

$$h_8 \Vdash^{\mathbf{P}} [g_8 \Vdash^{\mathbf{Q}} \tilde{\sigma} \in \mathscr{D}_0 \ \&\ \tilde{\sigma} \subseteq Y_0 \cup [0, f^4(1)) \ \&\ Y_0 \cup [0, f^4(1)) \in \mathscr{I}_{\mathscr{D}_0}].$$

Thus, $h_8 \Vdash^{\mathbf{P}} [g_8 \Vdash^{\mathbf{Q}} 0 = 1]$.

Thus, $\mathscr{D}$ cannot be extended to a P-point $\mathscr{D}_0$ in $(V^{\mathbf{P}})^{\mathbf{Q}}$. □

We now present the main result of the paper. This theorem was first proven by S. Shelah.

6.5. THEOREM [The Shelah P-point Independence Theorem]. *If* ZFC *is consistent, then*

"ZFC + *there are no P-points*" *is consistent.*

PROOF. Suppose not. $(V^P)^Q$ has a P-point $\mathscr{D}_0$. By 2.1, there exists a P-filter $\mathscr{D}_1 \in V^P$ such that $\mathscr{D}_1 \subseteq \mathscr{D}_0$. But this contradicts 6.4. □

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